Hierarchical Tangential Vector Finite Elements for Tetrahedra

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***Abstract*—Tangential vector finite elements (TVFE's) overcome most of the shortcomings of node-based finite elements for electromagnetic simulations. Hierarchical TVFE's are of considerable practical interest since they allow use of effective selective field expansions where different order TVFE's are combined within a computational domain. For a tetrahedral element, this letter proposes a set of hierarchical mixed-order TVFE's up to and including order 2.5 that differ from previously presented TVFE's. The hierarchical mixed-order TVFE's are constructed as the three-dimensional equivalent of hierarchical mixed-order TVFE's for a triangular element. They can be formulated for higher orders than 2.5, and the generalization to curved tetrahedral elements is straightforward.**

***Index Terms*—Finite-element method, hierarchical basis functions sets, higher order basis functions.**

I. Introduction

TANGENTIAL vector finite elements (TVFE's) based on expanding a vector field in terms of values associated with element edges have been shown to be free of the shortcomings of node-based finite elements [1]. TVFE's are therefore of considerable practical interest. Nédélec pointed out [2], [3] that it may not necessarily be advantageous to employ polynomial-complete TVFE's when applying the finite-element method (FEM). This leads to the introduction of attractive mixed-order TVFE's. A set of TVFE's is referred to as hierarchical if the vector basis functions forming the nth-order TVFE are a subset of the vector basis functions forming the $(n+1)$th-order TVFE, and this desirable property allows for effective selective field expansions combining different order TVFE's in different regions of the computational domain. For a large class of electromagnetic problems, hierarchical mixed-order TVFE's are therefore attractive for FEM discretization.

For a tetrahedral element, the lowest order TVFE was originally introduced by Whitney [4]. It provides a constant tangential/linear normal (CT/LN) field along element edges and a linear field at element faces and inside the element (complete to order 0.5). Mixed-order TVFE's providing a linear tangential/quadratic normal (LT/QN) field along element edges and a quadratic field at element faces and inside the element (complete to order 1.5) were presented by Lee *et al.* [5],

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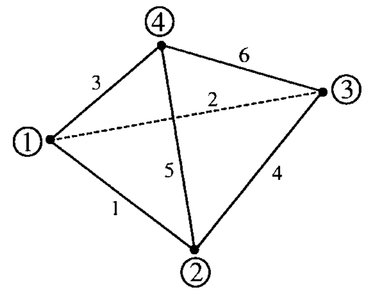

Fig. 1. Illustration of tetrahedral element and the numbering of nodes and edges.

Webb and Forghani [6], Savage and Peterson [7], and Graglia *et al.* [8]. Only the TVFE presented by Webb and Forghani compares to the Whitney TVFE in a hierarchical fashion. Nonhierarchical mixed-order TVFE's providing a quadratic tangential/cubic normal (QT/CuN) field along element edges and a cubic field at element faces and inside the element (complete to order 2.5) were presented by Savage and Peterson [7] (a correction to this TVFE was recently given by Peterson [9]) and Graglia *et al.* [8].

Hierarchical mixed-order TVFE's for a tetrahedral element have only been proposed up to and including order 1.5 [6], and these were written up by inspection. The purpose of this letter is to propose a set of hierarchical mixed-order TVFE's for a tetrahedral element beyond order 1.5. Specifically, hierarchical mixed-order TVFE's are presented up to and including order 2.5 where the mixed-order TVFE of order 1.5 differs from the one presented by Webb and Forghani [6]. We derive the hierarchical mixed-order TVFE's from existing nonhierarchical mixed-order TVFE's for a tetrahedral element [7], [9] and existing hierarchical mixed-order TVFE's for a triangular element [10], [11] in a systematic fashion that makes the proposed set of hierarchical mixed-order TVFE's for a tetrahedral element the direct three-dimensional (3-D) equivalent of the set of hierarchical mixed-order TVFE's for a triangular element [10], [11]. Hierarchical mixed-order TVFE's for higher orders than 2.5 can be derived by modifying the TVFE's proposed by Graglia *et al.* [8], and their extension to curved tetrahedral elements is straightforward via a simple mapping; see, for instance, [8].

II. Formulation

We consider a tetrahedral element with nodes 1, 2, 3, and 4 as shown in Fig. 1. The volume of the tetrahedron is denoted by V. Simplex (or volume) coordinates $\zeta_1, \zeta_2, \zeta_3$, and ζ_4 at a point P are defined in the usual manner, viz. $\zeta_n = V_n/V$ where V_n denotes the volume of the tetrahedron formed by P and the nodes of the triangular face opposite to node n. Below, vector basis functions will be formulated in terms of these coordinates. Vector basis functions associated with an edge or a face of the tetrahedron will be referred to as edge- or face-based vector basis functions, respectively. All other vector basis functions will be referred to as cell-based vector basis functions.

A mixed-order TVFE of order 0.5 providing CT/LN variation along element edges and linear variation at element faces and inside the element is characterized by six linearly independent vector basis functions. Whitney initially presented six such vector basis functions [4]. The 3-D equivalent of the two-dimensional (2-D) CT/LN vector basis functions presented in [10], [11] is identical to the vector basis functions presented by Whitney [4]. The six edge-based vector basis functions are[1]

$$\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i, \qquad i < j. \tag{1}$$

A mixed-order TVFE of order 1.5 providing LT/QN variation along element edges and quadratic variation at element faces and inside the element is characterized by 20 linearly independent vector basis functions. Savage and Peterson [7] proposed the 12 edge-based vector basis functions

$$\zeta_i \nabla \zeta_j, \qquad i \neq j \tag{2}$$

and the eight face-based vector basis functions

$$\left.\begin{array}{l} \zeta_k(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i) \\ \zeta_j(\zeta_k \nabla \zeta_i - \zeta_i \nabla \zeta_k) \end{array}\right\}, \qquad i < j < k. \tag{3}$$

The 20 linearly independent vector basis functions (2), (3) do not compare to the Whitney vector basis functions (1) in a hierarchical fashion. We propose to replace the 12 edge-based basis functions (2) by

$$\left.\begin{array}{r} \zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i \\ (\zeta_i - \zeta_j)(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i) \end{array}\right\}, \qquad i < j. \tag{4}$$

The 20 linearly independent vector basis functions (3)–(4) form a mixed-order TVFE of order 1.5 that compares hierarchically to the proposed mixed-order TVFE of order 0.5.

A mixed-order TVFE of order 2.5 providing QT/CuN variation along element edges and cubic variation at element faces and inside the element is characterized by 45 linearly independent vector basis functions. Savage and Peterson[2] [7], [9] proposed the 18 edge-based vector basis functions

$$\zeta_i(2\zeta_i - 1)\nabla \zeta_j, \qquad i \neq j \tag{5}$$

$$\zeta_i \zeta_j(\nabla \zeta_i - \nabla \zeta_j), \qquad i < j \tag{6}$$

[1] The vector basis functions presented in this letter are not normalized. Furthermore, the indexes i, j, and k in (1)–(12) are implicitly assumed to belong to the set $\{1, 2, 3, 4\}$.

[2] A correction of the QT/CuN vector basis functions initially proposed by Savage and Peterson [7] was given by Peterson [9]. This corrected set is the one presented here.

the 24 face-based vector basis functions

$$\left.\begin{array}{l} \zeta_k(2\zeta_k - 1)(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i) \\ \zeta_j(2\zeta_j - 1)(\zeta_k \nabla \zeta_i - \zeta_i \nabla \zeta_k) \end{array}\right\}, \qquad i < j < k \tag{7}$$

$$\nabla(\zeta_i \zeta_j \zeta_k), \qquad i < j < k \tag{8}$$

$$\zeta_k^2(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i), \qquad i \neq j \neq k \neq i \tag{9}$$

and the three cell-based vector basis functions

$$\begin{array}{c} \zeta_2\zeta_3(\zeta_1 \nabla \zeta_4 - \zeta_4 \nabla \zeta_1) \\ \zeta_2\zeta_4(\zeta_1 \nabla \zeta_3 - \zeta_3 \nabla \zeta_1) \\ \zeta_3\zeta_4(\zeta_1 \nabla \zeta_2 - \zeta_2 \nabla \zeta_1). \end{array} \tag{10}$$

The 45 linearly independent vector basis functions (5)–(10) do not compare to the Whitney vector basis functions (1) in a hierarchical fashion. We propose to replace the 18 edge-based basis functions (5)–(6) by

$$\left.\begin{array}{r} \zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i \\ (\zeta_i - \zeta_j)(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i) \\ (\zeta_i - \zeta_j)^2(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i) \end{array}\right\}, \qquad i < j. \tag{11}$$

Further, we propose to replace the eight face-based vector basis functions (7) by

$$\left.\begin{array}{l} \zeta_k(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i) \\ \zeta_j(\zeta_k \nabla \zeta_i - \zeta_i \nabla \zeta_k) \end{array}\right\}, \qquad i < j < k. \tag{12}$$

The 45 linearly independent vector basis functions (8)–(12) form a mixed-order TVFE of order 2.5 that compares hierarchically to the proposed mixed-order TVFE's of order 0.5 and 1.5.

The vector basis functions (1), (3)–(4), and (8)–(12) form a set of hierarchical mixed-order TVFE's of orders 0.5, 1.5, and 2.5, respectively. Such a set offers several advantages over nonhierarchical mixed-order TVFE's, especially for FEM solution of electromagnetic problems where the field varies nonuniformly over the computational domain. In such cases, a lower order TVFE can be employed in regions where the field varies smoothly whereas a higher order TVFE can be employed in regions where the field varies rapidly thus leading to an effective discretization of the unknown electromagnetic field.

III. Conclusion

For a tetrahedral element, we proposed a set of hierarchical mixed-order TVFE's up to and including order 2.5. These differ from previously presented TVFE's and were constructed as the 3-D equivalent of hierarchical mixed-order TVFE's for a triangular element. TVFE's for higher orders than 2.5 can be formulated in a similar manner and the generalization to curved tetrahedral elements is straightforward.

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